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LINEAR SYSTEMS

Diagnosis of Uncertain Systems with Arbitrary Relative Degree and Unknown Input

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Abstract—The problem of detecting sensor and actuator faults for a class of multivariable systems with arbitrary relative degree is considered. The proposed solution is based on the synthesis of unknown input observers. A modification is proposed that ensures a predefined convergence time, which is based on the dynamic regressor extension and mixing method and the Kreisselmeier scheme. The obtained solution is extended to the class of nonlinear systems with parametric uncertainties. Simulation results are provided to illustrate the effectiveness of the proposed approach.

Keywords: system diagnosis, fault detection, fault isolation, unknown input observer

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1. INTRODUCTION

The article addresses the problem of detecting sensor and actuator faults in nonlinear systems with arbitrary relative degree and parametric uncertainties under conditions of unmeasured input signal. There are two main approaches to solve the task of technical condition diagnosis: physical and analytical redundancy.

The first approach involves the integration of redundant sensors and actuators into the system. Despite its effectiveness, this method can lead to significant financial costs and face technological limitations. The second approach is based on the development of specialized observers [1]. This method uses mathematical models and measurable system data to detect faults without the need for additional equipment, thereby minimizing the limitations associated with physical redundancy.

Observer-based approaches demonstrate high effectiveness in detecting faults both at the inputs (actuators) and outputs (sensors) of the system [2–4]. The main idea is to analyze the differences between the measured outputs and their estimates (residuals). The use of structured observer sets, directional residual generators, or specialized filters helps to solve the problem of fault isolation. Structured observer sets are developed by designing specific generators. Each of them tuned to be sensitive only to a particular type of fault. Directed generators produce residual signal vectors in such a way that they change only in one direction in the residual space corresponding to a specific fault, allowing for precise fault identification. Special filters, based on observers, are designed to be sensitive to specific types of failures, increasing diagnostic accuracy. However, a significant drawback of these methods is their dependence on inaccuracies in the system's mathematical model, which can limit their applicability in cases of parametric and signal uncertainties.

Among observers, there is a special category known as unknown input observers (UIO) [5–8], which are designed to estimate the state vector when the input signal cannot be measured. This type of observer is particularly effective in the presence of various disturbances and noises in the control channel. The main idea of this approach is to create state vector observers that are insensitive to input signals.

In studies [9–15], numerous examples demonstrate the use of this class of observers for solving practical tasks such as fault detection, sensor and actuator diagnostics, as well as decentralized state vector estimation in applications like formation control of unmanned autonomous vehicles. These applications showcase the effectiveness of unknown input observers in various challenging operational conditions.

However the synthesis of such observers is possible only for systems where the relative degree (the difference between the degrees of the denominator and numerator of the transfer function) is equal to one. For systems with a higher relative degree, strict constraints are introduced. For example, studies [16, 17] require the measurability of the derivatives of the system's output signal. In [18, 19], the authors propose to split the original system dynamics equation to isolate the components of the state vector that can be observed based solely on the output.

This article presents a method for diagnosing sensor and actuator faults in dynamic multivariable systems with arbitrary relative degree. The novelty of the obtained results lies in the following:

- A method for detecting sensor and actuator faults in multivariable systems with arbitrary relative degree and unmeasurable input signal has been developed;
- A modification of the method ensuring predefined finite-time convergence has been proposed;
- The method has been extended to the class of nonlinear systems with parametric uncertainties and multi-harmonic disturbances.

The solution is based on the previously proposed by the authors method of unknown input observer synthesis [7]. Modifications have been developed to ensure the applicability of this approach to systems with parametric uncertainties, as well as a predefined finite-time convergence.

The article is structured as follows: Section 2 presents the problem statement; Section 3 is dedicated to the synthesis of the unknown input observer; Section 4 describes the method for detecting sensor and actuator faults; Section 5 introduces a modification of the observer that ensures finite-time convergence; Section 6 discusses the application of the developed method to nonlinear systems with parametric uncertainties; the results of computer simulations are presented in Section 7.

2. PROBLEM STATEMENT

Let us consider a multivariable linear time-invariant system

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + f_a(t)), \\ \qquad y(t) = Cx(t) + f_s(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input signal vector, $y(t) \in \mathbb{R}^l$ is the measurable output signal vector, A, B, C are known constant matrices of appropriate dimensions, $f_s(t)$ is the fault signal affecting the sensor measurements, and $f_a(t)$ is the fault signal affecting the system input. The system has a vector of relative degrees between the inputs and outputs $r = [r_1, \ldots, r_l]$.

The sensor fault signal $f_s(t)$ is an unknown vector function that acting on the sensor measurements during their failure. The fault signal $f_a(t)$ is affecting the system input and takes an unknown nonzero value when the actuators fail.

Let us introduce the following assumptions.

- The pairs of matrices A, B and A, C are controllable and observable, respectively.
- The matrices B and C have full column and row ranks, respectively [2].

It is necessary to develop a method that ensures the detection and isolation of sensor and actuator faults in the system (1). The solution is defined as a binary function of the following form:

$$J^i_{fault}(t) = \begin{cases} 1, & \text{if fault occurs,} \\ 0 & \text{else,} \end{cases}$$

where i is the index of the diagnosed actuator or sensor. To solve the problem, we will use state vector observers that are insensitive to the input signal. Additionally, we will extend the obtained results to the class of nonlinear systems with parametric uncertainties and multi-harmonic disturbances.

3. UNKNOWN INPUT OBSERVER SYNTHESIS

In this section, we present an algorithm for synthesis of an unknown input observer under the condition of no faults. Let us represent the matrices B and C as follows:

$$B = [B_1 \quad B_2 \quad \ldots \quad B_m], \quad C = [C_1^{\mathrm{T}} \quad C_2^{\mathrm{T}} \quad \ldots \quad C_l^{\mathrm{T}}]^{\mathrm{T}},$$

where B_i and C_i are the columns and rows of the corresponding matrices. Introduce into consideration

$$P = [C_1 A^{r_1} \quad C_2 A^{r_2} \quad \ldots \quad C_l A^{r_l}]^{\mathrm{T}},$$

$$N = \begin{bmatrix} C_1 A^{r_1-1} B_1 & C_1 A^{r_1-1} B_2 & \ldots & C_1 A^{r_1-1} B_m \\ C_2 A^{r_2-1} B_1 & C_2 A^{r_2-1} B_2 & \ldots & C_2 A^{r_2-1} B_m \\ \vdots & \vdots & \ddots & \vdots \\ C_l A^{r_l-1} B_1 & C_l A^{r_l-1} B_2 & \ldots & C_l A^{r_l-1} B_m \end{bmatrix}.$$

Consider unknown input observer

$$\dot{\hat{x}}(t) = M\hat{x}(t) + L(y(t) - C\hat{x}(t)) + Gy^{(r)}(t), \tag{2}$$

where $y^{(r)}(t) = [y_1^{(r_1)} \quad y_2^{(r_2)} \quad \ldots \quad y_l^{(r_l)}]$, $y_i^{(j)}$ is the jth derivative of the ith output of the system (1), and the matrices M, L, and G are the solutions of the system of equations.

$$\begin{cases} B - GN = 0, \\ M = A - GP, \\ F = M - LC, \end{cases} \tag{3}$$

where $G = B(N^{\mathrm{T}}N)^{-1}N^{\mathrm{T}}$. To clarify the procedure for selecting the matrix G, let us consider a simple academic example.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C^{\mathrm{T}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then the matrix $N = 1$ and $G^{\mathrm{T}} = [0 \quad 1]$.

Let us consider the observation error $\tilde{x}(t) = x(t) - \hat{x}(t)$. Differentiating it, taking into account (3), we obtain the dynamic model:

$$\begin{aligned}\dot{\tilde{x}}(t) &= Ax(t) + Bu(t) - M\hat{x}(t) - L(y(t) - C\hat{x}(t)) - Gy^{(r)} \\ &= Ax(t) + Bu(t) - M\hat{x}(t) + LC\hat{x}(t) - GPx(t) - GNu(t) - Ly(t) \\ &= (A - GP)x(t) - M\hat{x}(t) - LC\tilde{x}(t) \\ &= (M - LC)\tilde{x}(t) = F\tilde{x}(t).\end{aligned} \tag{4}$$

The matrix F defines the dynamics of the closed-loop system and is chosen by the developer, while L is calculated to ensure the required eigenvalues of the matrix F. It is obvious that if F is Hurwitz, the observation error converges to zero. This condition can be satisfied if the matrix pair (M, C) is observable, i.e., there exists a matrix L such that $M - LC$ is stable.

Let us define the necessary and sufficient conditions for the existence of a solution of the system of equations (3).

Theorem 1. *Let the control plant is described by the system of equations* (1), *and the observer* (2) *satisfies the following conditions:*

- $\operatorname{rank}(N) = \operatorname{rank}(B)$*;*
- *the matrix pair* (M, C) *is observable.*

Then the system of equations (3) *has a unique solution*

$$G = B(N^{\mathrm{T}}N)^{-1}N^{\mathrm{T}}.$$

Proof. The system of equations (3) has a solution if and only if $GN = B$. Let us rewrite it as follows:

$$N^{\mathrm{T}}G^{\mathrm{T}} = B^{\mathrm{T}}.$$

The matrix B^{T} belongs to the spectral space of N^{T}. Therefore,

$$\operatorname{rank}(B^{\mathrm{T}}) \leqslant \operatorname{rank}(N^{\mathrm{T}}) \Rightarrow \operatorname{rank}(B) \leqslant \operatorname{rank}(N).$$

On the other hand [20],

$$\operatorname{rank}(N) = \operatorname{rank}(PB) \leqslant \min\{\operatorname{rank}(P), \operatorname{rank}(B)\} \leqslant \operatorname{rank}(B).$$

Therefore, a solution to the system of equations (3) exists if and only if $\operatorname{rank}(N) = \operatorname{rank}(B)$. The solution for G is obtained as follows:

$$GN = B \Rightarrow GNN^{\mathrm{T}} = BN^{\mathrm{T}} \Rightarrow G = B(N^{\mathrm{T}}N)^{-1}N^{\mathrm{T}}.$$

Thus, the observer dynamics depend on the matrix F, defined in the last equation of system (3). It is easy to see that to ensure the stability of the matrix F, the matrix pair (M, C) must be observable. The theorem is proven.

The state vector observer (2) requires the measurability of unavailable derivatives of the output signal. Without loss of generality, let us assume that $r_1 \leqslant r_2 \leqslant \ldots \leqslant r_l$. To allow the construction of an observer for a system with relative degrees $r_i = 1$, $i = \overline{1, l}$, we introduce r_l auxiliary variables as follows:

$$
\begin{aligned}
&z_1(t) = \hat{x}(t) - Gy^{(r-1)}(t),\\
&\dot{z}_1(t) = F(z_1(t) + Gy^{(r-1)}(t)) + Ly(t),\\
&z_2(t) = z_1(t) - FGy^{(r-2)}(t),\\
&\dot{z}_2(t) = F(z_2(t) + FGy^{(r-2)}(t)) + Ly(t),\\
&z_3(t) = z_2(t) - F^2Gy^{(r-3)}(t), \qquad \ldots\\
&\dot{z}_{r_1}(t) = Fz_{r_1}(t) + F^{r_1}G\begin{bmatrix} y_1(t)\\ 0\\ \ldots\\ 0\end{bmatrix} + F^{r_1}G\begin{bmatrix} 0\\ y_2^{(r_2-r_1)}\\ \ldots\\ y_l^{(r_l-r_2)}\end{bmatrix} + Ly(t),\\
&\qquad\ldots\\
&\dot{z}_{r_l}(t) = Fz_{r_l}(t) + F^{r_1}G\begin{bmatrix} y_1(t)\\ 0\\ \ldots\\ 0\end{bmatrix} + F^{r_2}G\begin{bmatrix} 0\\ y_2(t)\\ \ldots\\ 0\end{bmatrix} + \ldots + F^{r_l}G\begin{bmatrix} 0\\ 0\\ \ldots\\ y_l(t)\end{bmatrix} + Ly(t).
\end{aligned}
$$

Using the auxiliary variables, the state vector estimate can be obtained as follows:

$$
\begin{cases}
\hat{x}(t) = z_1(t) + Gy^{(r-1)}(t)\\
\qquad = z_{r_l}(t) + F^{r_l-1}G\begin{bmatrix} 0\\ 0\\ \ldots\\ y_l(t)\end{bmatrix} + \ldots + F^{r_1}G\begin{bmatrix} 0\\ y_2^{(r_2-r_1-1)}(t)\\ \ldots\\ y_l^{(r_l-r_1-1)}(t)\end{bmatrix} + G\begin{bmatrix} y_1^{(r_1-1)}(t)\\ y_2^{(r_2-1)}(t)\\ \ldots\\ y_l^{(r_l-1)}(t)\end{bmatrix},\\
\dot{z}_{r_l}(t) = Fz_{r_l}(t) + F^{r_1}G\begin{bmatrix} y_1(t)\\ 0\\ \ldots\\ 0\end{bmatrix} + F^{r_2}G\begin{bmatrix} 0\\ y_2(t)\\ \ldots\\ 0\end{bmatrix} + \ldots + F^{r_l}G\begin{bmatrix} 0\\ 0\\ \ldots\\ y_l(t)\end{bmatrix} + Ly(t).
\end{cases}
\tag{5}
$$

For brevity, let us rewrite (5) as follows:

$$
\begin{cases}
\hat{x}(t) = W_1(z_{r_l}(t), y^{(r_l-1)}(t)),\\
\dot{z}_{r_l} = W_2(z_{r_l}(t), y(t)).
\end{cases}
\tag{6}
$$

The state vector observer (5) can be constructed for the estimation of the state vector in systems with relative degrees equal one. However, it can be used to solve the problem of fault detection in systems with arbitrary relative degrees by applying filters. This approach is presented in the next section.

4. FAULT DETECTION AND ISOLATION METHOD

Despite the limitations for synthesizing the observer (5), it can be effectively used for diagnosing systems with arbitrary relative degrees, particularly for detecting sensor and actuator faults.

4.1. Fault Detection

To solve the fault detection problem, we evaluate the difference between the measured output and its estimate obtained using the observer

$$J(t) = y(t) - C\hat{x}(t). \tag{7}$$

To eliminate the unmeasurable derivatives of the output signal, we apply a linear filter $\frac{\lambda^{r_l-1}}{(s+\lambda)^{r_l-1}}$ to (7), where $s = d/dt$, and λ is a positive constant:

$$J_f(t) = \frac{\lambda^{r_l-1}}{(s+\lambda)^{r_l-1}}[J(t)] = \frac{\lambda^{r_l-1}}{(s+\lambda)^{r_l-1}}[y(t)] - C\frac{\lambda^{r_l-1}}{(s+\lambda)^{r_l-1}}[W_1(z_{r_l}(t), y^{(r_l-1)}(t))], \tag{8}$$

where

$$\frac{\lambda^{r_l-1}}{(s+\lambda)^{r_l-1}}[W_1(z_{r_l}(t), y^{(r_l-1)}(t))] = \frac{\lambda^{r_l-1}}{(s+\lambda)^{r_l-1}}z_{r_l}(t) + F^{r_l-1}G\begin{bmatrix} 0 \\ 0 \\ \dots \\ \frac{\lambda^{r_l-1}}{(s+\lambda)^{r_l-1}}y_l(t) \end{bmatrix} + \dots$$

$$+F^{r_1}G\begin{bmatrix} 0 \\ \frac{\lambda^{r_l-1}s^{(r_2-r_1-1)}}{(s+\lambda)^{r_l-1}}y_2(t) \\ \dots \\ \frac{\lambda^{r_l-1}s^{(r_l-r_1-1)}}{(s+\lambda)^{r_l-1}}y_l(t) \end{bmatrix} + G\begin{bmatrix} \frac{\lambda^{r_l-1}s^{(r_1-1)}}{(s+\lambda)^{r_l-1}}y_1(t) \\ \frac{\lambda^{r_l-1}s^{(r_2-1)}}{(s+\lambda)^{r_l-1}}y_2(t) \\ \dots \\ \frac{\lambda^{r_l-1}s^{(r_l-1)}}{(s+\lambda)^{r_l-1}}y_l(t) \end{bmatrix}.$$

All signals in equation (8) are available for measurement. For example, instead of the unmeasurable derivative $y_l^{(r_l-1)}(t)$ in (7), the filtered output $\frac{(\lambda s)^{r_l-1}}{(s+\lambda)^{r_l-1}}y_l(t)$ is used in equation (8). If the fault signal is not a high-frequency oscillation, then for $||J(t)|| \neq 0$ we have $||J_f(t)|| \neq 0$. The required sensitivity to the fault signal frequencies is ensured by choosing the coefficient λ. We formulate the fault detection rule as the following expression:

$$J_{fault}(t) = \begin{cases} 1, & ||J_f(t)|| > \sigma, \\ 0, & ||J_f(t)|| \leqslant \sigma, \end{cases}$$

where $J_{fault}(t)$ is the fault indicator, and $\sigma > 0$ is the threshold value, either specified by the developer or adaptively adjusted during system operation to ensure robustness against measurement noise.

4.2. Sensor Fault Isolation

To isolate sensor faults, we represent the plant (1) in the form [2]:

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + f_a(t)), \\ y^j(t) = C^j x(t) + f_s^j(t), \\ y_j(t) = c_j x(t) + f_{sj}(t), \end{cases} \tag{9}$$

where $c_j \in \mathbb{R}^{1\times n}$ is the jth row of the matrix C, the matrix $C^j \in \mathbb{R}^{(l-1)\times n}$ is obtained by removing the jth row from the matrix C, and $y^j(t)$ is the vector $y(t)$ without the jth element. To diagnose the jth sensor, we construct an observer of the form (6) for the system described by the first two equations of (9):

$$\begin{cases} \hat{x}^j(t) = W_1(z^j_{r_l}(t), y^j(t)), \\ \dot{z}^j_{r_l} = W_2(z^j_{r_l}(t), y^j(t)), \end{cases} \tag{10}$$

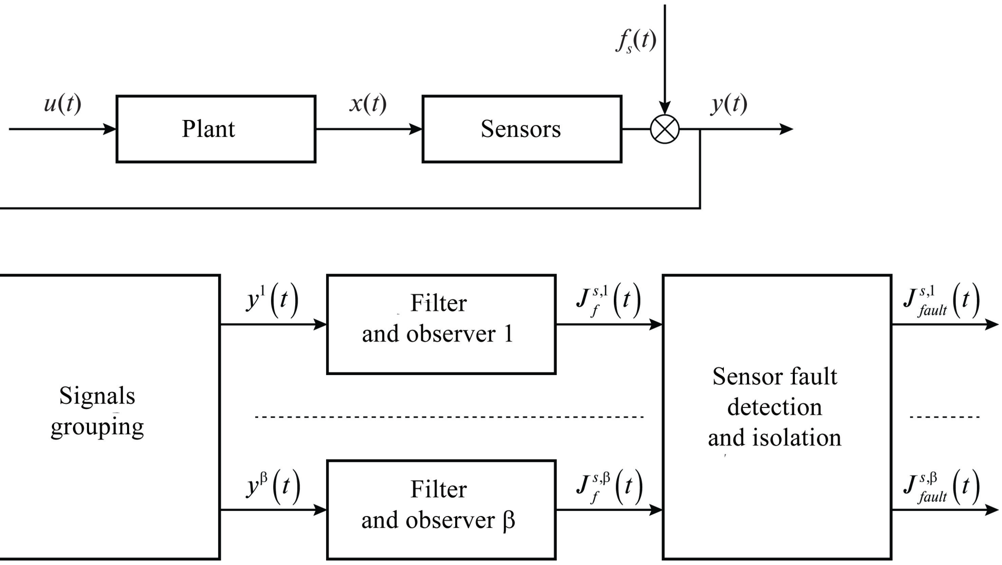

Fig. 1. Sensors fault detection and isolation scheme.

where $z_{r_l}^j(t)$ is the r_lth auxiliary variable for the system (9), and introduce the function

$$J_f^{s,j}(t) = \frac{\lambda^{r_l-1}}{(s+\lambda)^{r_l-1}} y^j(t) - C^j \frac{\lambda^{r_l-1}}{(s+\lambda)^{r_l-1}} W_1(z_{r_l}^j(t), y^j(t)).$$

The vector $J_f^{s,j}(t)$ is sensitive to the faults of all sensors except for the jth one. Thus, we can formulate the following rule for sensor fault isolation:

$$J_{fault}^{s,j}(t) = \begin{cases} 1, & ||J_f^{s,j}(t)|| < \sigma_{s,j}, \\ 0, & ||J_f^{s,j}(t)|| \geqslant \sigma_{s,k}, k = 1, \ldots, j-1, j+1, \ldots, l, \end{cases}$$

where $\sigma_{s,j}$ is the threshold value to ensure robustness against noise. The schematic of the sensor fault detection and isolation is shown in Fig. 1.

4.3. Actuator Fault Isolation

The approach outlined above cannot be applied for detecting actuator faults, as the observer is insensitive to the input signal. Therefore, we use a different approach to address this problem.

Based on Theorem 1, rewrite the first equation of the system (1), substituting the estimate of $x(t)$ obtained using (2). Considering (4), we have $x(t) = \hat{x}(t) + e^{Ft}\tilde{x}(0)$, thus the dynamic equation takes the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B(u(t) + f_a(t)) + \epsilon(t), \tag{11}$$

where $\epsilon(t)$ is an exponentially decaying function which can be neglected.

Apply a stable first-order linear filter to (11) and express the term containing the fault signal:

$$J^a(t) = \frac{\lambda}{s+\lambda}[Bf_a(t)] = \frac{\lambda s}{s+\lambda}[\hat{x}(t)] - \frac{\lambda}{s+\lambda}[A\hat{x}(t) + Bu(t)].$$

To eliminate the unmeasurable derivatives of the output signal in the observer (6), we also apply the filter $\frac{\lambda^{r_l-1}}{(s+\lambda)^{r_l-1}}$ and obtain a vector function, based on which we will perform actuator fault isolation:

$$J_f^a(t) = \frac{s\lambda^{r_l}}{(s+\lambda)^{r_l}} W_1\left(z_{r_l}(t), y^{(r_l-1)}\right) - \frac{\lambda^{r_l}}{(s+\lambda)^{r_l}}\left(AW_1(z_{r_l}(t), y^{(r_l-1)}) + Bu(t)\right). \quad (12)$$

In accordance with the problem statement, all columns of the matrix B are linearly independent. Therefore, in the case of a fault in the jth actuator, the vector $J_f^a(t)$ will be aligned with B_j. To ensure robustness and normalize the fault values of different actuators, we use the cosine of the angle between the vectors $J_f^a(t)$ and B_j. A larger cosine value corresponds to a smaller angle between the vectors being considered and, consequently, a higher probability of fault in the jth actuator. Thus, the fault isolation rule for the jth actuator will take the form

$$J_{fault}^{a,j}(t) = \begin{cases} 1, & \frac{||J_f^{a\,\mathrm{T}} B_j||}{||J_{a,f}||\,||B_j||} < \sigma_{a,j}, \\ 0, & \end{cases}$$

where $J_{fault}^{a,j}(t)$ takes the value of one in the event of a fault in the jth actuator and zero otherwise.

5. FINITE-TIME CONVERGENCE

The convergence time of $\hat{x}(t)$ to the true value depends on the eigenvalues of the matrix F. In several practical applications predefined finite-time convergence is required. To provide this capability, we will modify the observer (5).

Considering (4), we can express the state vector in terms of its estimate:

$$x(t) = \hat{x}(t) + e^{Ft}\tilde{x}(0). \quad (13)$$

It is obvious that an initial observation error is required to calculate the accurate value of the state vector. Let us substitute (6) into (13):

$$x(t) - W_1(z_{r_l}(t), y^{(r_l-1)}(t)) = e^{Ft}\tilde{x}(0),$$

multiply by the matrix C and apply the linear filter $\frac{\lambda^{r_l-1}}{(s+\lambda)^{r_l-1}}$ to eliminate the unmeasurable output derivatives:

$$y - CW_1(z_{r_l}(t), y^{(r_l-1)}(t)) = Ce^{Ft}\tilde{x}(0).$$

This equation can be represented in the form of linear regression

$$q(t) = m^{\mathrm{T}}(t)\tilde{x}(0), \quad (14)$$

where

$$q(t) = \frac{\lambda^{r_l-1}}{s+\lambda^{r_l-1}}\left(y(t) - CW_1(z_{r_l}(t), y^{(r_l-1)}(t))\right),$$
$$m^{\mathrm{T}}(t) = \frac{\lambda^{r_l-1}}{s+\lambda^{r_l-1}} Ce^{Ft}.$$

Let us define the initial observation error using (14), dynamic regressor extension and mixing method, and the Kreisselmeier scheme [21, 22]. Multiply (14) by $m(t)$:

$$m(t)q(t) = m(t)m^{\mathrm{T}}(t)\tilde{x}(0)$$

and apply a stable linear filter $H(s) = \frac{\lambda_f}{s+\lambda_f}$, where $\lambda_f > 0$, to obtain the extended linear regression

$$\begin{aligned} &Y(t) = \Phi(t)\tilde{x}(0), \\ &Y(t) = H(s)m(t)q(t), \\ &\Phi(t) = H(s)m(t)m^{\mathrm{T}}(t). \end{aligned} \tag{15}$$

State space representation of (15) takes the form

$$\begin{aligned} \dot{\Phi}(t) &= -\lambda_f\Phi(t) + m(t)m^{\mathrm{T}}(t), \quad \Phi(0) = 0, \\ \dot{Y}(t) &= -\lambda_f Y(t) + m(t)q(t), \quad Y(0) = 0. \end{aligned}$$

The aforementioned Kreisselmeier scheme allows for non-decaying excitation of the regressor for the time required to estimate the initial observation error.

In accordance with [21], we apply $n-1$ different stable linear filters $H_i(s) = \frac{\lambda_i}{s+\lambda_i}$ to (15) and multiply by the adjoint matrix $\Phi(t)$. The transformations performed allow us to obtain n scalar regression equations

$$\Upsilon_i(t) = \Delta(t)\tilde{x}_i(0), \quad i = \overline{1, n},$$

where $\Delta(t) = \det(\Phi(t))$, $\Upsilon_i(t) = \mathrm{Adj}(\Phi(t))Y_i(t)$, $\tilde{x}_i(0)$ is the ith element of $\tilde{x}(0)$.

The estimation of the initial error in the absence of noise can be obtained by a trivial solution.

$$\hat{\tilde{x}}_i(0) = \frac{\Upsilon_i(t)}{\max(\Delta(t), \varepsilon)}, \tag{16}$$

where ε is a small number to prevent division by zero during the initialization of the algorithm.

The influence of noise can be reduced by selecting an appropriate filter for (15), using low-pass filters, or moving average filters. We can formulate the following theorem.

Theorem 2. *If the matrix F has imaginary eigenvalues, which ensures the condition of non-decaying excitation of the signal $m^{\mathrm{T}}(t)$, then the equations* (13) *and* (14)–(16) *provide an estimate of $\tilde{x}(0)$ in a finite time.*

The proof of the theorem follows from the results presented in [22] and the calculations provided above. The convergence time is specified by the developer by choosing the time moment for the computation of (16).

This theorem and equation (13) provide an estimate of the state vector in finite-time required for the diagnostic of sensors and actuators.

6. APPLICATION TO NONLINEAR SYSTEMS WITH PARAMETRIC UNCERTAINTIES

Let us consider a nonlinear time-invariant system with parametric uncertainties

$$\begin{cases} \dot{x}(t) = Ax(t) + B[\Theta_u(u(t) + f_a(t)) + \Theta_x x(t) + \Theta_y\Phi_y(y,t) + \delta(t)], \\ y(t) = Cx(t) + f_s(t), \end{cases} \tag{17}$$

where

$$\Theta_u = \begin{bmatrix} \theta_{11u} & \theta_{12u} & \dots & \theta_{1mu} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{m1u} & \theta_{m2u} & \dots & \theta_{mmu} \end{bmatrix}, \quad \Theta_x = \begin{bmatrix} \theta_{11x} & \theta_{12x} & \dots & \theta_{1nx} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{m1x} & \theta_{m2x} & \dots & \theta_{mnx} \end{bmatrix},$$

$$\Theta_y = \begin{bmatrix} \theta_{11y} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \theta_{mmy} \end{bmatrix}, \quad \Phi_y(y,t) = \begin{bmatrix} \varphi_{1y}(y,t) \\ \vdots \\ \varphi_{my}(y,t) \end{bmatrix},$$

$\theta_{iju}, \theta_{ijx}, \theta_{iiy}$ are unknown parameters, φ_{iy} –, $\delta(t)$ is an external disturbance.

Let us introduce the assumption that the external disturbance $\delta(t)$ has the form $\sum\limits_{i=1}^{\bar{n}} R_i \sin(\omega_i t + \phi_i)$, where R_i, ω_i, and ϕ_i are the amplitudes, frequencies, and phases, respectively, and $\bar{n}$ is a known number of harmonics.

Since the observer (6) is robust with respect to input signals, the fault detection scheme for sensors, shown in Fig. 1, retains its functionality. However, uncertainties in the control channel prevent the application of the method for detecting actuator faults. To solve this problem, it is necessary to estimate the unknown parameters. Let us parameterize (17).

According to Theorem 2, the proposed observer provides an estimate of the state vector in a finite time. Substitute this estimate into (17):

$$\begin{gathered}\dot{\hat{x}}(t) = A\hat{x}(t) + B\bar{u}(t),\\ \dot{\bar{z}}_{r_l}(t) + F^{r_l-1}Gy_l^{(1)}(t) + \ldots + Gy^{(r)}(t) = A\left(\bar{z}_{r_l}(t) + F^{r_l-1}Gy_l(t) + \ldots + Gy^{(r-1)}(t)\right)\\ +B\left[\Theta_u u(t) + \Theta_x(\bar{z}_{r_l}(t) + F^{r_l-1}Gy_l(t) + \ldots + Gy^{(r-1)}(t)) + \Theta_y\Phi_y(y,t) + \delta(t)\right] + Bf_a^*(t),\end{gathered} \tag{18}$$

where new unknown input has the form

$$\begin{gathered}\bar{u}(t) = \Theta_u u(t) + \Theta_x\hat{x}(t) + \Theta_y\Phi_y(y,t) + \delta(t) + f_a^*(t)),\\ f_a^*(t) = \Theta_u f_a(t),\\ \bar{z}_{r_l}(t) = z_{r_l}(t) + e^{Ft}\hat{\bar{x}}(0).\end{gathered}$$

Assume that at the beginning of the plant's operation, there are no actuator faults, i.e., $Bf_a^*(t) = 0$. We apply a filter of order r_l, $\frac{\lambda_r^{r_l}}{(s+\lambda_r)^{r_l}}$, where $\lambda_r > 0$, to equation (18) and transform it into the form of linear regression

$$q_r(t) = B[m_r^{\mathrm{T}}(t)\Xi_r + \bar{\delta}(t)], \tag{19}$$

where

$$\begin{gathered}q_r(t) = \frac{\lambda_r^{r_l}}{(s+\lambda_r)^{r_l}}[\dot{\bar{z}}_{r_l}(t) + F^{r_l-1}Gy_l^{(1)}(t) + \ldots + Gy^{(r)}(t)]\\ -A(\bar{z}_{r_l}(t) + F^{r_l-1}Gy_l(t) + \ldots + Gy^{(r-1)}(t)),\end{gathered}$$

regressor with known signals

$$\begin{gathered}m_r(t) = [\; U_r(t) \quad S_r(t) \quad \Phi_r(t)\;]^{\mathrm{T}},\\ U_r(t) = \frac{\lambda_r^{r_l}}{(s+\lambda_r)^{r_l}}\left[u(t)\right],\\ S_r(t) = \frac{\lambda_r^{r_l}}{(s+\lambda_r)^{r_l}}\left[\bar{z}_{r_l}(t) + F^{r_l-1}Gy_l(t) + \ldots + Gy^{(r-1)}(t)\right],\\ \Phi_r(t) = \frac{\lambda_r^{r_l}}{(s+\lambda_r)^{r_l}}\left[\Phi_y(y,t)\right],\end{gathered}$$

vector of unknown parameters

$$\Xi_r = \left[\begin{array}{ccc}\Theta_u & \Theta_x & \Theta_y\end{array}\right]^{\mathrm{T}} \in \mathbb{R}^{\bar{m}},$$

filtered external disturbance

$$\bar{\delta}(t) = \frac{\lambda_r^{r_l}}{(s+\lambda_r)^{r_l}}\left[\delta(t)\right].$$

For simplicity of explanation, without loss of generality, let us assume that the disturbing signal consists of a single harmonic. Furthermore, relying on the property of the sinusoidal signal ($s^2[\sin \omega t] = -\omega^2 \sin \omega t$), rewrite (19) in the form

$$s^2 \left[q_r(t) - Bm_r^{\mathrm{T}}(t)\Xi_r \right] = -\omega^2 \left[q_r(t) - Bm_r^{\mathrm{T}}(t)\Xi_r \right], \tag{20}$$

where ω is a signal frequency.

To identify the unknown parameters in equation (20), only its first row is required. Multiply both sides of equation (20) by the matrix $\bar{B}$ such that $\bar{B}B = 1$, and by applying a second-order stable linear filter $\frac{\lambda_\delta^2}{(s+\lambda_\delta)^2}$, we obtain

$$q_\delta(t) = m_\delta^{\mathrm{T}}(t)\Xi_\delta, \tag{21}$$

where

$$m_\delta^{\mathrm{T}}(t) = \left[\frac{s^2\lambda_\delta^2}{(s+\lambda_\delta)^2} \left[m_r^{\mathrm{T}}(t) \right], \frac{\lambda_\delta^2}{(s+\lambda_\delta)^2} \left[-\bar{B}q_r(t) \right], \frac{\lambda_\delta^2}{(s+\lambda_\delta)^2} \left[m_r^{\mathrm{T}}(t) \right] \right],$$

$$\Xi_\delta = \begin{bmatrix} \Xi_r & \omega^2 & \omega^2\Xi_r \end{bmatrix}^{\mathrm{T}},$$

$$q_\delta(t) = \frac{s^2\lambda_\delta^2}{(s+\lambda_\delta)^2} \left[\bar{B}q_r(t) \right].$$

It should be noted that if the signal $\bar{\delta}(t)$ contains multiple harmonics, the system (19) can also be transformed into the form of linear regression [23].

To estimate the unknown parameters in the linear regression equation (21), we apply the FT DREM identification method described in (14)–(16) [19, 20]. Furthermore, the filtered external disturbance can be easily computed as follows:

$$\bar{\delta}_r(t) = \bar{B}q_r(t) - m_r^{\mathrm{T}}(t)\hat{\Xi}_r = \theta_{\sin}\sin(\hat{\omega}t) + \theta_{\cos}\cos(\hat{\omega}t) = m_{\bar{\delta}}^{\mathrm{T}}(t)\theta_{\bar{\delta}}, \tag{22}$$

where $m_{\bar{\delta}}^{\mathrm{T}}(t) = \begin{bmatrix} \sin(\hat{\omega}t) & \cos(\hat{\omega}t) \end{bmatrix}$, $\theta_{\bar{\delta}} = \begin{bmatrix} \theta_{\sin} \\ \theta_{\cos} \end{bmatrix}$, $\theta_{\sin}$, $\theta_{\cos}$ are amplitudes of filtered external disturbance.

It is evident that equation (22) takes the form of a linear regression equation and any identification method can be applied to estimate the amplitude of the external disturbance.

As a result, all unknown parameters of the original system have been identified, allowing for the application of the actuator fault detection method described in (11) and (12).

Thus, the generalized algorithm for a system with parametric uncertainties can be described by the following steps:

1. Construct the unknown input observers (5).
2. Use the unknown input observers and the method described in (7)–(10) for sensor fault detection.
3. Based on functioning without faults sensors, construct the unknown input observers and parameterize the original system in the form of (17).
4. Estimate the unknown parameters of the original plant using the method described in (18)–(22).
5. Perform diagnosis of actuator faults.

Remark 1. It should be noted that the state vector is estimated based on the outputs of the original plant. This means that at least one output must have a reliable value. That is, it is necessary to construct at least two unknown input observers for sensor fault detection. On the other hand, for actuator fault detection, the proposed solution requires the construction of only one output observer, allowing the problem to be solved even with a single input signal.

7. COMPUTER SIMULATION

Perform computer simulation to demonstrate the effectiveness of the proposed method. Let us consider the following nonlinear plant with parametric uncertainties:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = k_1 x_1(t) + k_2 x_2(t) + k_3 x_3(t) + k_4 x_3(t) + k_5(u(t) + f_a(t)) + k_6 \sin(y_1), \\ \dot{x}_3(t) = x_4(t), \\ \dot{x}_4(t) = x_1(t) - 2x_3(t), \\ y_1(t) = x_1(t) + \upsilon_1(t), \\ y_2(t) = x_3(t) + f_s(t) + \upsilon_2(t), \\ y_3(t) = x_4(t) + \upsilon_3(t), \end{cases} \tag{23}$$

where $k_1, k_2, k_3, k_4, k_5, k_6$ are unknown parameters, $f_s(t)$ is the sensor fault signal, $f_a(t)$ is the actuator fault signal, and $\upsilon_1(t)$, $\upsilon_2(t)$, $\upsilon_3(t)$ are measurement noises with a Gaussian distribution(mean is 0.005 and the variance is 0.005).

Rewrite (23) in the state space representation corresponding to the problem statement (17):

$$\begin{cases} \dot{x}(t) = Ax(t) + B\Big[\begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix} x(t) + \theta_5(u(t) + f_a(t)) + \theta_6 \sin(y_1)\Big], \\ y(t) = Cx(t) + C_s f_s(t) + \upsilon(t), \end{cases} \tag{24}$$

where $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $C_s = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $u(t) = 1 + 2\sin(t)$, $\upsilon(t) = [\upsilon_1(t), \upsilon_2(t), \upsilon_3(t)]^{\mathrm{T}}$. The equations (24) contain unknown parameters related to the plant parameters $\theta_1 = k_1 + 4 = -1$, $\theta_2 = k_2 + 3 = 3$, $\theta_3 = k_3 - 2 = -3$, $\theta_4 = k_4 = 1$, $\theta_5 = k_5 = 2$, $\theta_6 = k_6 = -2$. It should be noted that when constructing the diagnostic system, the coefficients θ_i are assumed to be unknown and are determined during the estimation process.

Define the fault signals as follows. The fault signal of the second sensor $f_s(t)$ is zero until the moment $t = 5$ seconds, after which it takes a non-zero value of $3 + \cos(t)$. The actuator fault occurs at the 15th second, and its signal takes the value $f_a(t) = 5$.

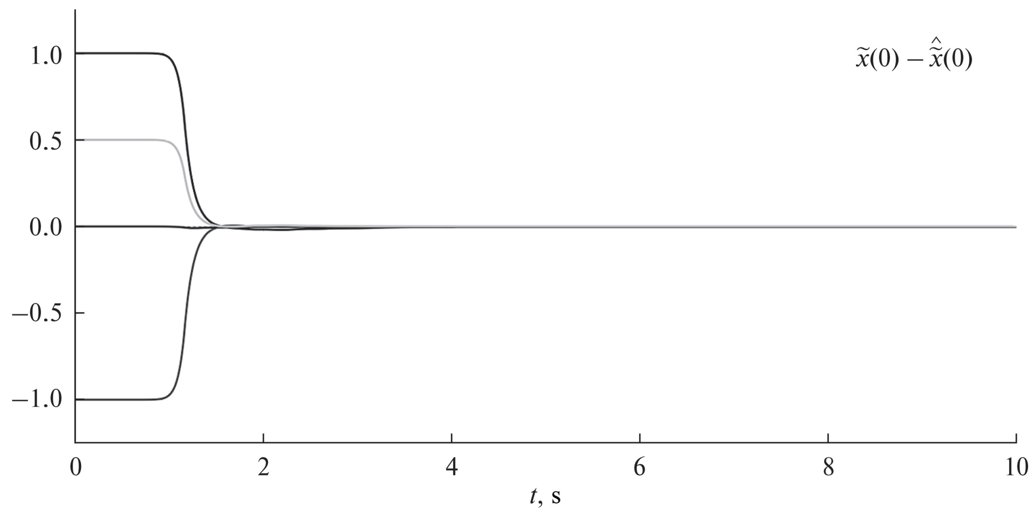

Fig. 2. Initial conditions estimation error.

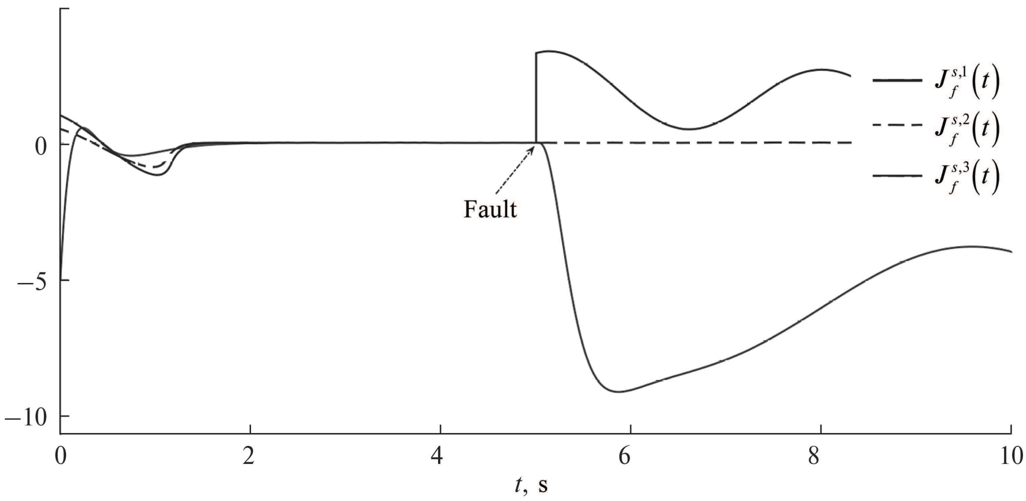

Fig. 3. Residual in the case second sensor fault.

The plant (24) has relative degrees $r_1(y_1) = 2$, $r_2(y_2) = 4$, $r_3(y_3) = 3$. Construct three observers using the auxiliary variables (5) for $\hat{x}^i(t)$, where $i = 1, 2, 3$. The matrices L^i are chosen such that the matrices $F^i = M^i - L^iC^i$ are Hurwitz orhave eigenvalues lying on the imaginary axis, when applying the algorithm with finite convergence time. The matrices $C^i \in \mathbb{R}^{(l-1)\times n}$ are obtained by removing the ith row from the matrix C.

To compare the observers with asymptotic and finite-time convergence, the first observer was synthesized with the Hurwitz matrix F^1, while the matrices F^2 and F^3 have imaginary eigenvalues:

$$L^1 = place\left((M^1)^{\mathrm{T}}, (C^1)^{\mathrm{T}}, \begin{bmatrix} -4 & -5 & -6 & -7 \end{bmatrix}\right)^{\mathrm{T}}, \quad F^1 = \begin{bmatrix} 0 & 1 & 0 & -76 \\ 2 & 0 & 0 & -150 \\ 0 & 0 & -7 & 0 \\ 1 & 0 & 0 & -15 \end{bmatrix},$$

$$L^2 = \begin{bmatrix} 0 & 5 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \quad F^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix},$$

$$L^3 = \begin{bmatrix} 0 & 5 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}^{\mathrm{T}}, \quad F^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix}.$$

The parameters of the estimation algorithms are defined as follows: $\frac{\lambda_0}{s+\lambda_0} = \frac{\lambda_1}{s+\lambda_1} = \ldots = \frac{\lambda_\delta}{s+\lambda_\delta} = \frac{5}{s+5}$, $a_0 = a_1 = \ldots = a_\delta = 0.5$, $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_\delta = 10^{-7}$, threshold value $\sigma = 5$.

Figures 2–5 show the results of the simulation. The transient processes of the initial condition estimation errors are presented in Fig. 2. It is evident that these signals converge to zero. Figure 3 shows the residual signals during the fault of the second sensor. These signals converge to zero after the initial condition estimation. When the sensor fault occurs, $J_f^{s,2}(t)$ continues to maintain a zero value, while $J_f^{s,1}(t)$ and $J_f^{s,3}(t)$ deviate, indicating correct isolation of the fault in the second sensor. Comparing the results of the observers $J_f^{s,1}(t)$ and $J_f^{s,3}(t)$, it is important to note that using the observer with finite-time convergence allows for quicker fault detection. Based on the sensors not affected by faults, an unknown input observer is formed for actuator diagnostics.

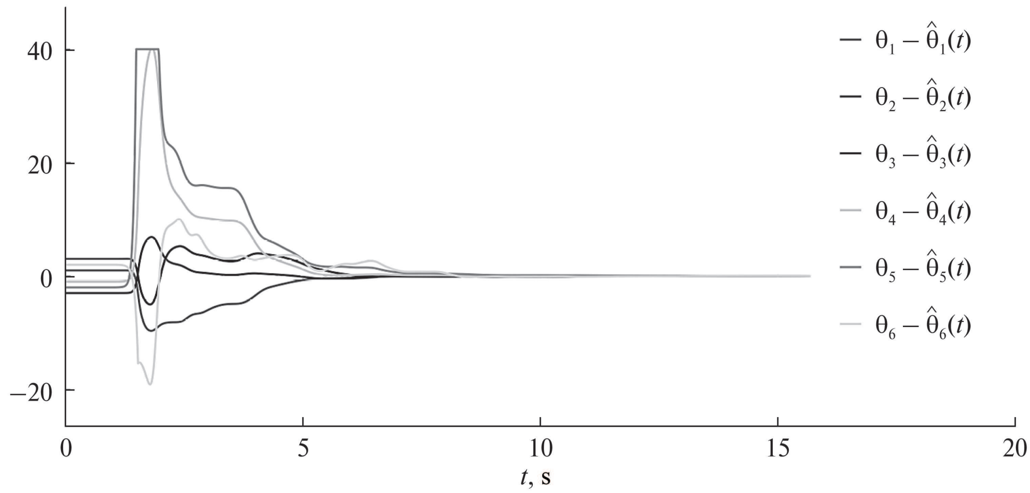

Fig. 4. Unknown parameters estimation error.

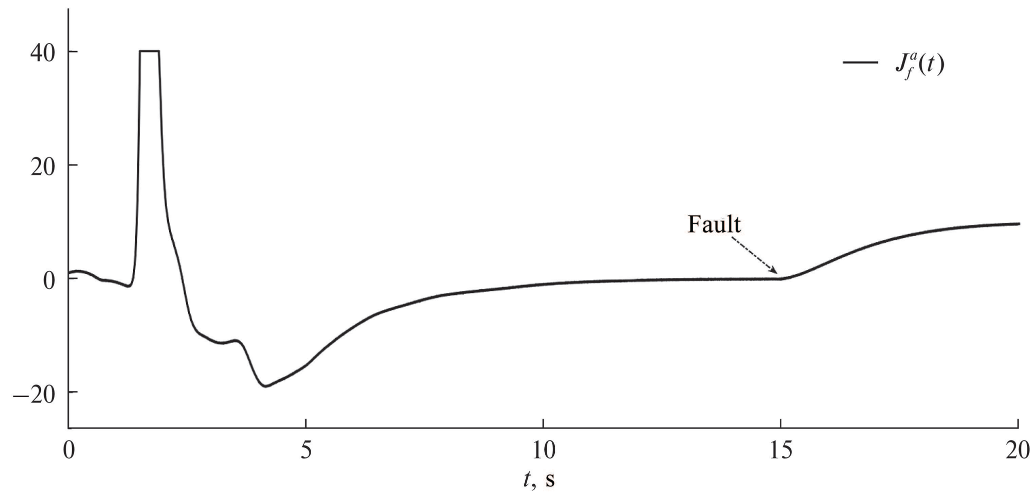

Fig. 5. Residual in the case of actuator fault.

In Fig. 4 the transients of the estimation of the unknown parameters are presented. The actuator fault estimation signal is shown in Fig. 5. This signal converges to zero as a result of parameter identification and takes a non-zero value after 15 seconds, indicating correct fault isolation.

A threshold value σ is used for the fault detection algorithm to ensure robustness against noise. The use of filters also helps to reduce the impact of measurement noise on the diagnostic method. However, filtering inevitably contributes a delay in the diagnostic signals.

8. CONCLUSION

This paper presents a method for diagnosing sensors and actuators in multivariable linear systems with arbitrary relative degree. The proposed approach is based on observers that are invariant to the input signal. Modifications for the developed observer are proposed to ensure predefined finite-time convergence and applicability to systems with parametric uncertainties and multi-harmonic disturbances. Computer simulations were conducted to confirm the effectiveness and functionality of the proposed approach. The obtained results can also be extended to the class of non-stationary systems, where the parameters are outputs of linear generators.

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